DICKSON INVARIANTS IN THE IMAGE OF THE STEENROD SQUARE

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Abstract. Let D_n be the Dickson invariant ring of $\mathbb{F}_2[X_1, \ldots, X_n]$ acted by the general linear group $\mathrm{GL}(n, \mathbb{F}_2)$. In this paper, we provide an elementary proof of the conjecture by [3]: each element in D_n is in the image of the Steenrod square in $\mathbb{F}_2[X_1, \ldots, X_n]$, where $n > 3$.

1. Introduction

A polynomial in $\mathbb{F}_2[X_1, X_2, \ldots, X_n]$ is *hit* if it is in the image of the summation of the Steenrod square: $\sum_{i\geq 1} \mathrm{Sq}^i$. Let D_n be the Dickson invariant algebra of n-variables. In this paper, we will prove the following,

Theorem 1.1. *When $n > 3$, each polynomial in the Dickson invariant ring D_n is hit.*

In [3], Hung studies the Dickson invariants in the image of the Steenrod square. Since it is trivial that D_1 and D_2 are not hit, the problem starts interesting from $n = 3$. In the same paper, Hung shows that each element in D_3 is hit and conjectured that it is true for $D_{n>3}$. So our result provides a positive answer to the conjecture, which supports to the positive answer of the conjecture on the spherical classes: there are no spherical classes in Q_0S^0, except the Hopf invariant one and Kervaire invariant one elements. We refer to [3] and an excellent expository paper [5], p501 for more background regarding to this conjecture.

Remark 1.2. Recently, K. F. Tan and the author [4] has obtained an elementary proof of the case $n = 3$.

2. Proof of Theorem 1.1

We first recall some basic properties regarding the Dickson algebra.

Write V_n for the product

$$\prod_{\alpha_i\in\{0,1\}, i=1,..,n-1} (\alpha_1 x_1 + \cdots + \alpha_{n-1}x_{n-1} + x_n).$$

Then we have the following theorem.

Theorem 2.1 (Hung [2])**.**

$$\mathrm{Sq}^i V_n = \begin{cases} V_n \text{ if } i = 0 \\ V_n Q_{n-1,s} \text{ if } i = 2^{n-1} - 2^s,\ 0 \leq s \leq n-1 \\ V_n^2 \text{ if } i = 2^{n-1} \\ 0 \text{ otherwise.} \end{cases}$$

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$$\mathrm{Sq}^i Q_{n,s} = \begin{cases} Q_{n,r} \text{ if } i = 2^s - 2^r,\ r \le s \\ Q_{n,r}Q_{n,t} \text{ if } i = 2^n - 2^t + 2^s - 2^r,\ r \le s < t \\ Q_{n,s}^2 \text{ if } i = 2^n - 2^s \\ 0 \text{ otherwise.} \end{cases}$$

In the following, we will frequently use the above results without mentioning each time.

We use the induction on n to prove Theorem 1.1. Suppose that the statement is true for n. Then we will prove that each polynomial in D_{n+1} is hit.

Recall that

$$Q_{n+1,k} = Q_{n,k-1}^2 + V_{n+1}Q_{n,k} \quad \text{for } 1 \le k \le n.$$

So any monomial in $\mathbb{F}_2[Q_{n+1,0}, Q_{n+1,1}, ..., Q_{n+1,n}]$ can be written as the summation of the following form:

$$A := V_{n+1}^a Q_{n,0}^{n_0} Q_{n,1}^{n_1} Q_{n,2}^{n_2} \cdots Q_{n,n-1}^{n_{n-1}}.$$

Hence by the hypothesis of the induction, it is sufficient to show that A is hit for any $a > 0$. Notice that

$$V_{n+1} = \sum_{s=1}^{n} \mathrm{Sq}^1(Q_{n,s}X_{n+1}^{2^s-1}). \tag{1}$$

When n_1 is even, we have the hit polynomial

$$A = \mathrm{Sq}^1[\left(\sum_{s=1}^{n} Q_{n,s}x_{n+1}^{2^s-1}\right) V_{n+1}^{a-1} Q_{n,0}^{n_0} Q_{n,1}^{n_1} Q_{n,2}^{n_2} \cdots Q_{n,n-1}^{n_{n-1}}].$$

If n_1 is odd and n_2 is even, then A can be written as the hit polynomial:

$$\begin{aligned} &\mathrm{Sq}^2[V_{n+1}^a Q_{n,0}^{n_0} Q_{n,1}^{n_1-1} Q_{n,2}^{n_2+1} \cdots Q_{n,n-1}^{n_{n-1}}] \\ &\qquad + \mathrm{Sq}^1[\left(\sum_{s=1}^{n} Q_{n,s}x_{n+1}^{2^s-1}\right) V_{n+1}^{a-1} Q_{n,0}^{n_0} (\mathrm{Sq}^1 Q_{n,1}^{\frac{n_1-1}{2}})^2 Q_{n,2}^{n_2+1} \cdots Q_{n,n-1}^{n_{n-1}}] \end{aligned}$$

In the following, we will always assume that n_1 and n_2 are both odd.

When $n = 3$, n_0 is even and a is odd, we have

$$\begin{aligned} A &= (V_4^{a-1}\mathrm{Sq}^4 V_4) Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2-1} \\ &\equiv V_4 \chi(\mathrm{Sq}^4)[V_4^{a-1} Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2-1}] \text{ (modulo the hits)} \\ &\equiv V_4^a Q_{3,1} \left(\mathrm{Sq}^2[Q_{3,0}^{\frac{n_0}{2}} Q_{3,1}^{\frac{n_1-1}{2}} Q_{3,2}^{\frac{n_2-1}{2}}]\right)^2 \text{ (modulo the hits).} \end{aligned}$$

Using the previous observation, the last polynomial is hit, since the order of $Q_{3,2}$ is even.

When $n = 3$, n_0 is even and a is even, notice that

$$Q_{3,0}^{n_0} Q_{3,1}^{n_1} Q_{3,2}^{n_2} = Q_{3,0}^{n_0} Q_{3,1}^{n_1-1} Q_{3,2}^{n_2-1} \mathrm{Sq}^4 Q_{3,1}.$$

Then using the χ-*trick* and doing some basic computation, we can see that the monomial $Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2}$ is in the image of $\sum_{i=1}^{4}\mathrm{Sq}^i$. In fact,

$$\begin{aligned}
&Q_{3,1}\chi(\mathrm{Sq}^4)[Q_{3,0}^{n_0}Q_{3,1}^{n_1-1}Q_{3,2}^{n_2-1}]\\
&= [\mathrm{Sq}^2 Q_{3,2}][\mathrm{Sq}^2(Q_{3,0}^{\frac{n_0}{2}}Q_{3,1}^{\frac{n_1-1}{2}}Q_{3,2}^{\frac{n_2-1}{2}})]^2\\
&\equiv Q_{3,2}\chi(\mathrm{Sq}^2)[Q_{3,0}^{\frac{n_0}{2}}Q_{3,1}^{\frac{n_1-1}{2}}Q_{3,2}^{\frac{n_2-1}{2}}]^2 \text{ (modulo the hits)}\\
&= (Q_{2,1}^2+V_3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{\frac{n_0}{2}}Q_{3,1}^{\frac{n_1-1}{2}}Q_{3,2}^{\frac{n_2-1}{2}})]\\
&= \mathrm{Sq}^2\left(Q_{2,1}[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{\frac{n_0}{2}}Q_{3,1}^{\frac{n_1-1}{2}}Q_{3,2}^{\frac{n_2-1}{2}})^2]\right)\\
&+\mathrm{Sq}^1\left\{(Q_{2,1}X_3+X_3^3)[\mathrm{Sq}^1\mathrm{Sq}^2(Q_{3,0}^{\frac{n_0}{2}}Q_{3,1}^{\frac{n_1-1}{2}}Q_{3,2}^{\frac{n_2-1}{2}})]^2\right\},
\end{aligned}$$

where we have used (1) in the last equality. On the other hand, $\mathrm{Sq}^iV_4^a = 0$ for $i = 1, 2, 3$ and 4. Therefore using the χ-*trick* , we know that A is hit.

When $n = 3$, n_0 is odd and a is odd, the polynomial A equals

$$V_4^{a-1}(\mathrm{Sq}^7V_4)Q_{3,0}^{n_0-1}Q_{3,1}^{n_1}Q_{3,2}^{n_2}.$$

From the discussion above, we know that

$$Q_{3,0}^{n_0-1}Q_{3,1}^{n_1}Q_{3,2}^{n_2}$$

is in the image of $\sum_{i=1}^{4}\mathrm{Sq}^i$. On the other hand, $\chi(\mathrm{Sq}^i)(V_4^{a-1}\mathrm{Sq}^7V_4) = 0$ for $i = 1, 2, 3$ and 4. So using the χ-*trick* , we conclude that A is hit.

When $n = 3$, n_0 is odd and a is even, let ν be the integer such that $a = 2^\nu b$ where b is odd. Then

$$V_4^a = \mathrm{Sq}^{4a}\mathrm{Sq}^{2a}\cdots\mathrm{Sq}^{8b}V_4^b.$$

Hence

$$\begin{aligned}
A &= (\mathrm{Sq}^{4a}\mathrm{Sq}^{2a}\cdots\mathrm{Sq}^{8b}V_4^b)(Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2})\\
&\equiv V_4^b\chi(\mathrm{Sq}^{8b})\cdots\chi(\mathrm{Sq}^{2a})\chi(\mathrm{Sq}^{4a})(Q_{3,0}^{n_0}Q_{3,1}^{n_1}Q_{3,2}^{n_2}) \quad \text{(modulo the hits)}.
\end{aligned}$$

After expanding the last polynomial using Theorem 2.1, it is easy to see that each resulting term belongs to one of the previous cases. Therefore A is hit.

When $n \geq 4$, the polynomial A takes the following form,

$$V_{n+1}^a(\mathrm{Sq}^{2^n-4}Q_{n,1})Q_{n,0}^{n_0}Q_{n,1}^{n_1-1}Q_{n,2}^{n_2-1}\cdots Q_{n,n-1}^{n_{n-1}}. \tag{2}$$

Using a result of Don Davis, Theorem 2. of [1] and the χ-*trick* , we know that it is sufficient to show the polynomial:

$$Q_{n,1}\mathrm{Sq}^{2^{n-1}}\cdots\mathrm{Sq}^8\chi(\mathrm{Sq}^4)\left\{V_{n+1}^aQ_{n,0}^{n_0}Q_{n,1}^{n_1-1}Q_{n,2}^{n_2-1}\cdots Q_{n,n-1}^{n_{n-1}}\right\}$$

is hit.

After expansion using the Steenrod operation, the above polynomial can be written as the summation of the form:

$$V_{n+1}^aQ_{n,0}^{k_0}Q_{n,1}^{k_1}Q_{n,2}^{k_2}\cdots Q_{n,n-1}^{k_{n-1}}.$$

Using the previous discussion, we can conclude that all these polynomials are hit, except for those when k_1 and k_2 are both odd. But in this case, we can replace n_i by k_i for all i in (2) and carry out the above process again. After using this process sufficiently many times with modulo the hits, we can conclude that the new k_0, k_1

and k_2 are independent of the process. To keep k_0, k_1 and k_2 unchanged with the process, we must require that

$$\mathrm{Sq}^{2^{n-1}} \cdots \mathrm{Sq}^8 \chi(\mathrm{Sq}^4) \left\{ V_{n+1}^a Q_{n,3}^{k_3} \cdots Q_{n,n-1}^{k_{n-1}} \right\} \text{ (modulo the hits)}$$

contributes $Q_{n,2}$ after each process is done, since for $j \leq 2^{n-1}$ and $t < n$, $\mathrm{Sq}^j Q_{n,0} = Q_{n,0}Q_{n,t}$ only if $j = t = n-1 (> 2)$. Finally because all k_i $(0 \leq i < n)$ are finite, we conclude that A is hit after carrying on the process further for enough many times.

References

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